ISRAEL JOURNAL OF MATHEMATICS **131** (2002), 139–147

A NEW BANACH SPACE WITH VALDIVIA DUAL UNIT BALL

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ABSTRACT

We give an example of a Banach space which admits no projectional resolution of the identity but whose dual unit ball in weak* topology is a Valdivia compact. This answers a question asked by M. Fabian, G. Godefroy and V. Zizler.

1. Introduction

Projectional resolutions of the identity are a powerful tool in studying the structure of non-separable Banach spaces. They were introduced by J. Lindenstrauss [L1], [L2] in the 1960s. Their importance became clear after the famous paper [AL], where it is proved that each weakly compactly generated Banach space admits a projectional resolution. Later, this result was extended to several larger classes of spaces. These classes can be defined by topological properties of the dual unit ball equipped with the weak* topology. So, if the dual unit ball of X is an Eberlein compactum or, more generally, a Corson compactum, X admits a projectional resolution [V1], [V3]. Since [AMN], a generalization of Corson compacta has been studied [V2], [V3]; this class was given the name of Valdivia compacta in [DG]. The natural question arose — whether a Banach space with

* Partially supported by Research grants GAUK 277/2001, GAUK 160/1999 and MSM 113200007.
Received March 1, 2001 and in revised form October 12, 2001

Valdivia dual unit ball admits a projectional resolution. This question was formulated in [FGZ, Remark 2 on p. 224], later in [K3, Question 1] or in [K4, Question 4.21]. Some partial positive results were given already in [V2] for $C(K)$ spaces with K Valdivia and in [V3] for some other Banach spaces. In the present paper we give a counterexample.

Let us start with definitions.

Definition 1: Let X be a Banach space of the density $\kappa > \aleph_0$. A **projectional resolution of the identity** (PRI) on X is an indexed family $(P_\alpha \mid \omega \leq \alpha \leq \kappa)$ of projections on X with the following properties.

(i) $P_\omega = 0$, $P_\kappa = \mathrm{Id}_X$;
(ii) $\|P_\alpha\| = 1$ for $\omega < \alpha \leq \kappa$;
(iii) $\operatorname{dens} P_\alpha X \leq \operatorname{card} \alpha$ for $\omega < \alpha \leq \kappa$;
(iv) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \kappa$;
(v) $P_\alpha X = \overline{\bigcup_{\beta<\alpha} P_\beta X}$ if $\alpha \leq \kappa$ is limit.

The existence of a PRI is an isometric notion, due to the condition (ii). It may happen that a Banach space has a PRI with respect to one equivalent norm but has no PRI with respect to another equivalent norm — see, e.g., [DGZ, p. 259], [FGZ, Examples 1 and 2] or [K3]. So it is useful to define, following [PY], an isomorphic analogue of PRI.

Definition 2: Let X be a Banach space of the density $\kappa > \aleph_0$. A **bounded projectional resolution** (BPR) on X is an indexed family $(P_\alpha \mid \omega \leq \alpha \leq \kappa)$ of projections on X which satisfies all properties of a PRI except for condition (ii), which is replaced by the following.

(ii′) $\sup_{\omega\leq\alpha\leq\kappa} \|P_\alpha\| < \infty$.

The supremum is called the **projection constant** of the BPR.

Now we are going to define Valdivia compacta and related notions.

Definition 3:

(1) If Γ is a set, we put

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is countable}\}.$$

Let K be a compact Hausdorff space.

(2) We say that $A \subset K$ is a **Σ-subset of** K if there is a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$.
(3) K is called a **Corson compact space** if K is a Σ-subset of itself.
(4) K is called a **Valdivia compact space** if K has a dense Σ-subset.

Before defining the Banach space analogues of these notions, let us fix the following concept.

Definition 4: Let X be a Banach space, $S \subset X^*$ and $C \geq 1$. We say that S is **C-norming**, if

$$\frac{1}{C}\|x\| \leq \sup\{|\xi(x)|: \xi \in S \cap B_{X^*}\} \leq \|x\|$$

for every $x \in X$. S is called **norming** if it is C-norming for some $C \geq 1$.

Let us remark that it follows from the Hahn–Banach separation theorem that a linear subspace $S \subset X^*$ is C-norming if and only if

$$\frac{1}{C}B_{X^*} \subset \overline{S \cap B_{X^*}}^{w^*} \subset B_{X^*}.$$

Definition 5:

(1) Let X be a Banach space. We say that $S \subset X^*$ is a **Σ-subspace** of X^* if there is a linear one-to-one weak* continuous mapping $T: X^* \to \mathbb{R}^\Gamma$ such that $S = T^{-1}(\Sigma(\Gamma))$.

(2) A Banach space X is called **weakly Lindelöf determined** (WLD) if X^* is a Σ-subspace of itself.

(3) A Banach space X is called **C-Plichko** (where $C \geq 1$) if X^* has a C-norming Σ-subspace. X is called **Plichko** if it is C-Plichko for some $C \geq 1$.

It follows from [V3] that any 1-Plichko space admits a PRI. In [FGZ, Lemma 2] it is proved that a Banach space X with density $\aleph_1$ admits a PRI if and only if it is 1-Plichko. Finally, by [K2], see [K4, Theorem 4.15], a Banach space X is 1-Plichko if and only if the dual unit ball (B_{X^*}, w^*) has a dense convex symmetric Σ-subset. Therefore X admits a PRI whenever (B_{X^*}, w^*) has a dense convex symmetric Σ-subset. We will show that convexity cannot be omitted.

2. Main result

Our main result is the following theorem.

THEOREM: *There is a Banach space X, isomorphic to $C[0, \omega_1]$, which admits no projectional resolution of the identity but whose dual unit ball is a Valdivia compactum.*

This theorem settles the isometric question of the existence of PRI. But the isomorphic one remains open since X has a PRI in some equivalent norm (as it is isomorphic to $C[0, \omega_1]$). This question and related problems will be discussed in the final section.

3. Auxilliary results

We begin with the following general lemma on convexity of certain functions.

LEMMA 1: *Let X be a linear space, A a convex subset of X and $f\colon A \to \mathbb{R}$ a function continuous on each segment in A such that $|f|$ is convex. Then the function $F\colon A \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,y) = |f(x)| + |y - f(x)|$ is convex.*

Proof: We will prove the statement in several steps.

STEP 1: F is convex on $A \times \mathbb{R}$ if (and only if) it is convex on each segment. Hence, without loss of generality, we can assume that $X = \mathbb{R}$ and $A = [a,b]$ is a closed interval.

STEP 2: We will use the following elementary fact. Let $p < q < r$ be real numbers, h a continuous function on $[p,r]$ which is convex on $[p,q)$ and on $(q,r]$ and, moreover, $h'_-(q) \le h'_+(q)$. Then h is convex on $[p,r]$.

STEP 3: The functions f^+ and f^- are convex on $[a,b]$.

If $f \ge 0$ or $f \le 0$ on the whole $[a,b]$, then the claim is trivial. Otherwise there is, by convexity of $|f|$ and continuity of f, some $c \in (a,b)$ such that $f \le 0$ on $[a,c)$ and $f \ge 0$ on $(c,b]$ (or vice versa). Then

$$f^+(x) = \begin{cases} 0 & x \in [a,c) \\ |f(x)| & x \in [c,b] \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} |f(x)| & x \in [a,c), \\ 0 & x \in [c,b]. \end{cases}$$

Now, by Step 2 these functions are convex.

STEP 4: Let

$$G^+ = \{(x,y) \in [a,b] \times \mathbb{R}\colon y \ge f(x)\}$$

and

$$G^- = \{(x,y) \in [a,b] \times \mathbb{R}\colon y \le f(x)\}.$$

Then F is convex on each segment contained either in G^+ or in G^-.

On G^+ we have $F(x,y) = |f(x)| + y - f(x) = 2f^-(x) + y$. On the other set $F(x,y) = 2f^+(x) - y$. So the assertion follows by Step 3.

STEP 5: Let $[P,R]$ be a segment in $[a,b] \times \mathbb{R}$ such that there is $Q \in (P,R)$ with $[P,Q) \subset G^+$ and $(Q,R] \subset G^-$. Then F is convex on $[P,Q]$.

By Step 4 the function F is convex both on $[P,Q)$ and on $(Q,R]$. Further, put $F_1(x,y) = |f(x)|$ and $F_2(x,y) = |y - f(x)|$. We have $F_2(Q) = 0$, so F_2 has in Q a local minimum, hence the left-sided derivative of F_2 in Q in the direction $R - P$ is non-positive while the right-sided one is non-negative. The function F_1

is convex, hence the left-sided derivative of F_1 in Q in the direction $R-P$ is less than or equal to the right-sided one. Taking the sum and using Step 2 we can conclude that F is convex on $[P,R]$.

STEP 6: For any two points $P,R \in [a,b]\times\mathbb{R}$ there are finitely many points $Q_1,\dots,Q_k$ lying on the segment $[P,R]$ such that each of the segments $[P,Q_1]$, $[Q_1,Q_2],\dots,[Q_k,R]$ lies either in G^+ or in G^-.

Put $\tilde{F}_2(x,y)=y-f(x)$. It is enough to show that $\phi\colon t\mapsto \tilde{F}_2(P+t(R-P))$ is piecewise monotone. To check this it suffices to observe that

$$\phi'_+(t)=(r_2-p_2)-f'_+(p_1+t(r_1-p_1))(r_1-p_1)$$

(where, of course, $P=(p_1,p_2)$ and $R=(r_1,r_2)$) is piecewise monotone as f'_+ is piecewise monotone (cf. Step 3).

STEP 7: It follows from Step 5 and Step 6 that F is convex on each closed segment lying in $[a,b]\times\mathbb{R}$. Therefore that F is convex on $[a,b]\times\mathbb{R}$. This completes the proof. ∎

Now we fix some notation which we will use in the remaining lemmas of this section. By E we will denote the Banach space $C[0,\omega_1]$ equipped with the max-norm, by M its dual space, represented by finite signed Radon measures on $[0,\omega_1]$ equipped with the norm of total variation and the weak* topology, and by B_M the unit ball of M. Finally, let $f\colon M\to\mathbb{R}$ be a fixed weak* continuous function such that $f(0)=0$ and put

$$A(f)=\{\mu\in B_M\colon \mu(\{\omega_1\})=f(\mu)\},$$
$$B(f)=\{\mu\in M\colon |\mu|[0,\omega_1)+|f(\mu)|+|\mu(\{\omega_1\})-f(\mu)|\le 1\}.$$

LEMMA 2: *For any $\mu\in M$ there is $\alpha<\omega_1$ such that $f(\mu)=f(\nu)$ whenever $\nu\in M$ is such that $\nu\restriction[0,\alpha]=\mu\restriction[0,\alpha]$ and $\nu(\alpha,\omega_1]=\mu(\alpha,\omega_1]$.*

Proof: Let $\mu\in M$ be arbitrary. Then $f^{-1}(f(\mu))$ is a weak* G_δ set containing μ. Hence the statement follows easily from the definition of the weak* topology. ∎

LEMMA 3: *There is some $\delta>0$ such that $\delta B_M\subset B(f)\subset B_M$.*

Proof: The inclusion $B(f)\subset B_M$ follows from the triangle inequality. Indeed, if $\mu\in B(f)$, then

$$\|\mu\|=|\mu|[0,\omega_1)+|\mu(\{\omega_1\})|\le|\mu|[0,\omega_1)+|f(\mu)|+|\mu(\{\omega_1\})-f(\mu)|\le 1.$$

Further, as f is weak* continuous, it is also norm-continuous, and so the function $\mu \mapsto \|\mu\| + 2|f(\mu)|$ is norm-continuous. Hence there is $\delta > 0$ such that $\|\mu\| + 2|f(\mu)| \leq 1$ whenever $\|\mu\| \leq \delta$. Therefore for $\mu \in \delta B_M$ we have

$$|\mu|[0,\omega_1) + |f(\mu)| + |\mu(\{\omega_1\}) - f(\mu)| \leq \|\mu\| + 2|f(\mu)| \leq 1,$$

thus $\mu \in B(f)$. ∎

LEMMA 4: *The set $B(f)$ is the weak* closure of $A(f)$. Moreover, $A(f)$ is a Σ-subset of $B(f)$, so $B(f)$ is a Valdivia compactum.*

Proof: It is clear from the definitions that $A(f) \subset B(f)$. Let us first prove that $A(f)$ is weak* dense in $B(f)$. Let $\mu \in B(f)$ and $\alpha < \omega_1$ be the ordinal from Lemma 2. For $\gamma \in (\alpha, \omega_1)$ put

$$\mu_\gamma = \mu \restriction [0,\omega_1) + (\mu(\{\omega_1\}) - f(\mu)) \cdot \delta_\gamma + f(\mu) \cdot \delta_{\omega_1}.$$

It is clear that the net μ_γ weak* converges to μ. Further, $\|\mu_\gamma\| \leq 1$ by the definition of $B(f)$. Finally, by the choice of α we have $f(\mu_\gamma) = f(\mu) = \mu_\gamma(\{\omega_1\})$, hence $\mu_\gamma \in A(f)$.

Next we will show that $B(f)$ is weak* closed. Let μ_τ be a net of elements of $B(f)$ weak* converging to some $\mu \in M$. Let $\alpha < \omega_1$ be such that $\mu \restriction (\alpha, \omega_1) = 0$. Then

$$\begin{aligned} 1 &\geq \liminf_\tau (|\mu_\tau|[0,\omega_1) + |f(\mu_\tau)| + |\mu_\tau(\{\omega_1\}) - f(\mu_\tau)|) \\ &\geq \liminf_\tau |\mu_\tau|[0,\alpha] + \liminf_\tau (|\mu_\tau|(\alpha,\omega_1) + |f(\mu_\tau)| + |\mu_\tau(\{\omega_1\}) - f(\mu_\tau)|) \\ &\geq |\mu|[0,\alpha] + |f(\mu)| + \liminf_\tau (|\mu_\tau(\alpha,\omega_1)| + |\mu_\tau(\{\omega_1\}) - f(\mu_\tau)|) \\ &\geq |\mu|[0,\alpha] + |f(\mu)| + \liminf_\tau |\mu_\tau(\alpha,\omega_1] - f(\mu_\tau)| \\ &= |\mu|[0,\alpha] + |f(\mu)| + |\mu(\alpha,\omega_1] - f(\mu)| \\ &= |\mu|[0,\omega_1) + |f(\mu)| + |\mu(\{\omega_1\}) - f(\mu)|, \end{aligned}$$

hence $\mu \in B(f)$. It follows that $B(f)$ is weak* closed and therefore weak* compact (due to Lemma 3).

Finally, to show that $A(f)$ is a Σ-subset of $B(f)$ let us consider the mapping $h\colon B(f) \to \mathbb{R}^{[-1,\omega_1]}$ defined by the formula

$$h(\mu)(\alpha) = \begin{cases} \mu(\alpha,\omega_1] - f(\mu), & \alpha < \omega_1, \\ f(\mu), & \alpha = \omega_1. \end{cases}$$

It is clear that the mapping h is weak* continuous and one-to-one, and that $A(f) = h^{-1}(\Sigma([-1,\omega_1]))$. ∎

LEMMA 5: *Suppose that, moreover, f is odd and $|f|$ is convex. Then $B(f)$ is convex and symmetric, so it is a dual unit ball of an equivalent norm on E. However, $A(f)$ is convex if and only if f is affine on $B(f)$.*

Proof: If f is odd, then $B(f)$ is clearly symmetric. If $|f|$ is convex, then $B(f)$ is convex by Lemma 1. Hence $B(f)$ is convex and symmetric. If f is affine on $B(f)$, then clearly $A(f)$ is convex. Conversely, if $A(f)$ is convex, then f is affine on $A(f)$ (as $f(\mu) = \mu(\{\omega_1\})$ and $\mu \mapsto \mu(\{\omega_1\})$ is affine on $A(f)$), and so it is affine on $B(f)$ by Lemma 4 due to weak* continuity of f. ∎

4. Proof of the main result

Take, say, $f(\mu) = \mu(\{0\})^3$ or $f(\mu) = \mu(\{0\}) \cdot |\mu(\{0\})|$. Then, due to Lemma 5, $B(f)$ is a dual unit ball of an equivalent norm $|\cdot|$ on E and it is Valdivia by Lemma 4. Moreover, $A(f)$ is a dense Σ-subset of $B(f)$ by Lemma 4 and it is not convex by Lemma 5. As $B(f)$ has a dense set of G_δ points, $A(f)$ is the only dense Σ-subset of $B(f)$. This follows from [K4, Corollary 1.12] or [K2, Lemma 4.2], or already from [K1, Lemma 2.3 and Proposition 2.4]. We recall here the simple argument for the sake of completeness.

Let A' be another dense Σ-subset of $B(f)$. Both A' and $A(f)$ are dense and countably compact, hence they contain all G_δ points of $B(f)$. Thus $A' \cap A(f)$ is dense in $B(f)$. Let $\mu \in A(f)$. Then $\mu \in \overline{A' \cap A(f)}$, hence there is a sequence $\mu_n \in A' \cap A(f)$ such that $\mu_n \to \mu$. (This follows from the well-known fact that $\Sigma(\Gamma)$ is a Fréchet–Urysohn space; see [N, Theorem 2.1] or [K4, Lemma 1.6].) As A' is clearly sequentially closed, we have $\mu \in A'$. Therefore $A(f) \subset A'$. By interchanging the roles of A' and $A(f)$ we get $A' = A(f)$.

So $B(f)$ has no convex dense Σ-subset, hence $(E, |\cdot|)$ is not 1-Plichko. Finally, by [FGZ, Lemma 2], this space has no PRI. This completes the proof.

5. Final remarks and open problems

We proved that there is a Banach space with Valdivia dual unit ball without PRI. But as this space is isomorphic to $C[0, \omega_1]$, it has a BPR. In fact, it is Plichko. So the following question seems to be interesting.

QUESTION 1: *Is there a Banach space with Valdivia dual unit ball which admits no bounded projectional resolution?*

As ℓ_1-sums preserve Banach spaces with Valdivia dual unit ball (this follows from [K1, Theorem 4.1], see also [K4, Theorem 3.29]) but not Banach spaces with a BPR, see [PY, Section 7], the following question seems to be a natural step to answer Question 1.

QUESTION 2: *Let $C > 1$ be arbitrary. Is there a Banach space X with Valdivia dual unit ball such that any bounded projectional resolution on X has projection constant greater than C?*

One can ask whether we can get such examples by a refinement of our construction. By some elementary computations one can prove the following.

THEOREM′: *There is a Banach space X and $C > 1$ with the following properties.*

1. *X is isomorphic to the space $C[0, \omega_1]$.*
2. *The dual unit ball of X is a Valdivia compactum in its weak* topology.*
3. *Any bounded projectional resolution in X has projection constant greater than C.*
4. *X admits a bounded projectional resolution with projection constant at most* 3.

We can take $X = (E, |\cdot|)$ where $B_{X^*} = B(f)$ with $f(\mu) = |\mu(\{0\})| \cdot \mu(\{0\})$ and any $C < (15 - 3\sqrt{5})/8$. It is possible that by choosing another function f we could obtain a better C. However, it can be easily checked that our method necessarily yields $C < 3$.

Another question is whether an analogous example can be found within $C(K)$ spaces.

QUESTION 3: *Is there a compact Hausdorff space K such that $C(K)$ admits no PRI but the dual unit ball of $C(K)$ is a Valdivia compactum?*

If such a K exists, it cannot have a dense set of G_δ points — see [K2, Theorem 4.10] or [K4, Theorem 5.3]. Therefore none of our counterexamples is isometric to a $C(K)$ space. (Our spaces are isomorphic to $C[0, \omega_1]$, hence they are Asplund and thus K would be scattered [HHZ, Theorem 296].)

ACKNOWLEDGEMENT: The author is grateful to Matias Raja and Luděk Zajíček for helpful discussions on Lemma 1.

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